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A theory for scattering by density fluctuations based on three-wave interaction

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The theory of scattering by charged particle density fluctuations of a plasma is developed for the case of zero magnetic field. The source current is derived on the basis of, first, a three-wave interaction between the incident and scattered electromagnetic waves and one electrostatic plasma wave (either Langmuir or ion-acoustic), and secondly, a synchronous interaction between the same two electromagnetic waves and the discrete components of the charged particle fluctuations. Previous work is generalized by no longer making the assumption that the frequency of the electromagnetic waves is large compared to the plasma frequency. The general result is then applied to incoherent scatter, and to scatter by strongly driven plasma waves. An expansion is carried out for each of these cases to determine the lower order corrections to the usual high-frequency scattering formulae.

1. Introduction

The scattering of electromagnetic waves by density fluctuations has been a topic of general interest for many years. The first derivations, given by Booker (1955) and Villars & Weisskopf (1955), were based on the idea that density fluctuations give rise to dipole-moment density fluctuations which in turn cause the familiar far-field electric dipole radiation. Most studies since then on scattering use the same basic idea. Rosenbluth & Rostoker (1962) used a technique based on a far-field expansion of Maxwell's equations, and a source current consisting of a summation over discrete plasma particles. Birmingham *et al.* (1965), although not specifically addressing themselves to the far-field problem, showed that this scattering formula must be corrected by a factor equal to the refractive index of the scattered wave. All of these theories are based on the assumption that the incident and scattered electromagnetic waves are much higher in frequency than the plasma frequency, and lead to the result that the scattered power is proportional to the spectral density of the electron density fluctuations.

When the density fluctuations are excited by the random motion of charged particles, the scattering is referred to as incoherent scatter. The study of incoherent scattering of electromagnetic waves by a plasma has been given by a number of authors. Dougherty & Farley (1960), Salpeter (1960) and Fejer (1960), independently calculated the cross-section for random thermal fluctuations of the electron density. Hagfors (1961) extended the theory to include a static magnetic field. Rosenbluth & Rostoker (1962) generalized the theory to take into account departure from thermal equilibrium. The subject of scattering by density variations, and in particular incoherent scattering, was thoroughly reviewed by Bekefi (1966).

In this paper we extend this earlier work. The far field is first determined in terms of an asymptotic expansion of Maxwell's equations (Lighthill 1960), then the effects of synchronous interactions are evaluated by solving the Vlasov equation to second order in the electric field and using the result to calculate the source currents to second order. One obtains thereby a general theory for the scattering by density fluctuations for an arbitrary spectrum in E^2, $f_u E$ and f_u^2 (where E is the electric field, and f_u is that component of the electron velocity distribution function arising from the discrete nature of charged particles). The theory is valid for any ratio of the frequency of the electromagnetic waves to the plasma frequency, subject, of course, to the condition that the frequency is not so close to a resonance that multiple scattering effects must be included.

In the case of incoherent scatter, we carry the problem forward to a final solution. We obtain expressions for E^2, $f_u E$ and f_u^2, and obtain a closed-form solution in terms of unperturbed particle velocity distribution functions. The general theory is also applied to the case where the plasma waves are so strongly driven by an external source that one can neglect the effects of the random motions of the charged particles.

The results for both the case of arbitrary spectra and for incoherent scatter show the important result that, in general, the scattered power is no longer simply proportional to the spectral density of the electron density fluctuations, as is normally assumed, but that a more subtle dependence is involved.

Since the expressions for the scattered power are somewhat involved, an expansion is carried out, both for the case of incoherent scatter and strongly driven waves, in inverse powers of the frequency of the incident electromagnetic wave. We recover the results of the workers mentioned above and demonstrate, thereby, that our main results are quite general and encompass previous work as a subcase.

Our study has been carried out under the assumption that the static magnetic field is zero, that the charged particle velocity distribution functions are isotropic in velocity space, and that the medium is homogeneous.

We conclude § 1 by reviewing some of the fundamentals of the scattering process. The source currents responsible for the scattering are determined on the basis of two types of interaction, one depending on collective effects and one on discrete particle effects. These two effects arise, in turn, from the fact that the charged particle distribution function may be resolved into two components. One is the spatially averaged part associated with plasma waves and collective effects, and the second is the spatially rapidly fluctuating component which vanishes when averaged over the macroscopic volume. It arises from the discrete motion of the particles and is basically a thermal fluctuation phenomenon.

As is well known, the mechanism for the collective source current is basically no more than a three-wave plasma interaction between the incident and scattered electromagnetic waves on one hand, and a scattering electrostatic plasma wave

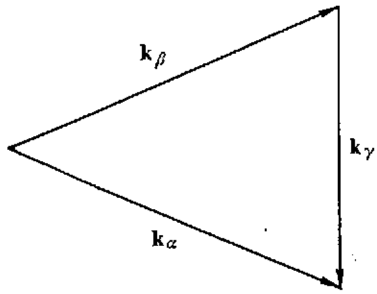

Fig. 1

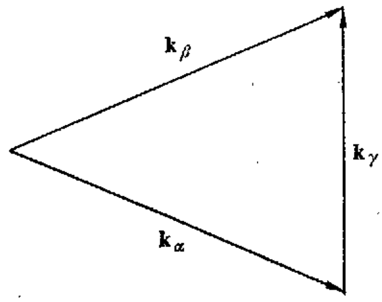

Fig. 2

FIGURE 1. Mixing of an incoming transverse wave $\mathbf{k}_\beta$ and an electrostatic wave $\mathbf{k}_\gamma$ to produce a scattered transverse wave $\mathbf{k}_\alpha$.

FIGURE 2. Decay of an incoming transverse wave $\mathbf{k}_\beta$ into an electrostatic wave $\mathbf{k}_\gamma$ and a scattered transverse wave $\mathbf{k}_\alpha$.

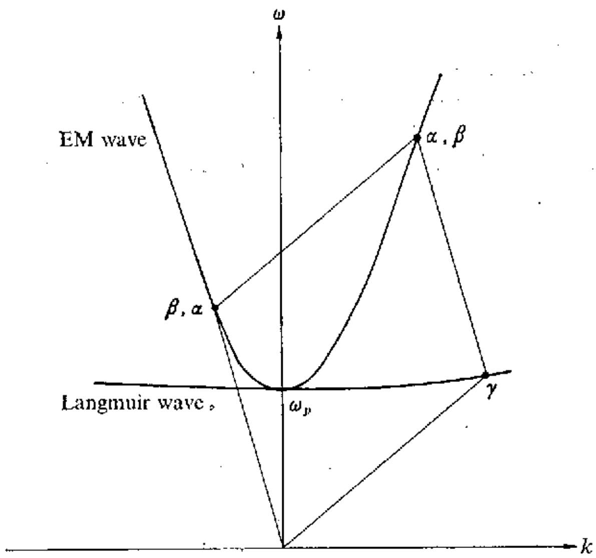

FIGURE 3. Synchronism diagram for the interaction of two transverse waves and a Langmuir wave.

on the other hand. The plasma wave may be either a Langmuir or ion-acoustic wave. A schematic of the process is shown in figures 1 and 2. In figure 1, the incoming wave $(\omega_\beta, \mathbf{k}_\beta)$ mixes with the electrostatic plasma wave $(\omega_\gamma, \mathbf{k}_\gamma)$ to produce a scattered electromagnetic wave $(\omega_\alpha = \omega_\beta + \omega_\gamma, \mathbf{k}_\alpha = \mathbf{k}_\beta + \mathbf{k}_\gamma)$. In the second version of the process, shown in figure 2, the incoming electromagnetic wave $(\omega_\beta, \mathbf{k}_\beta)$ decays into an electrostatic plasma wave $(\omega_\gamma, \mathbf{k}_\gamma)$ and a scattered electromagnetic wave $(\omega_\alpha = \omega_\beta - \omega_\gamma, \mathbf{k}_\alpha = \mathbf{k}_\beta - \mathbf{k}_\gamma)$. A synchronism diagram showing the dispersion curves of the interacting waves and the synchronism parallelogram for the conditions $\omega_\alpha = \omega_\beta \pm \omega_\gamma$, $\mathbf{k}_\alpha = \mathbf{k}_\beta \pm \mathbf{k}_\gamma$ corresponding to figures 1 and 2, respectively, is shown in figures 3 and 4 for the case where the electrostatic wave is a Langmuir wave and an ion-acoustic wave, respectively.

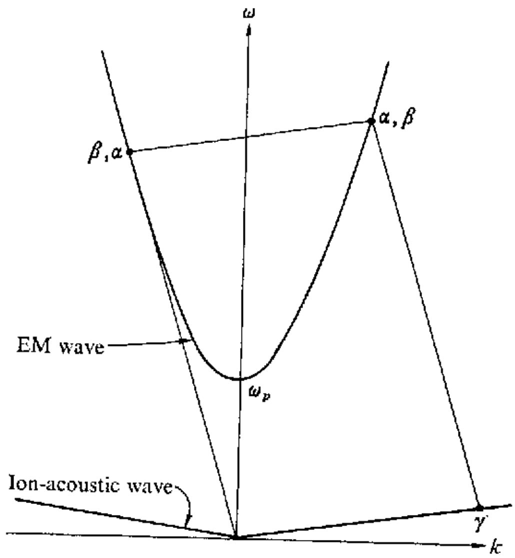

FIGURE 4. Synchronism diagram for the interaction of two transverse waves and an ion-acoustic wave.

The mechanism for the source current arising from discrete particle effects is an interaction between the electromagnetic waves again, and the synchronous Fourier component of the fluctuating discrete component of the electron velocity distribution functions. This source current is responsible for scattering by unscreened electrons, i.e. scattering which does not involve collective effects between the particles.

2. Theory for scattering in terms of current sources

In §2 we consider the scattering in general, without specifying the current sources responsible. Our system is described by Maxwell's equations

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \tag{1}$$

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}^{(1)} + \mathbf{J}^{(2)}, \tag{2}$$

and the Vlasov equation

$$\frac{\partial f}{\partial t} + (\mathbf{v}.\nabla) f + \eta(\mathbf{E} + \mathbf{v} \times \mathbf{B}).\frac{\partial f}{\partial \mathbf{v}} = 0, \tag{3}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, $\mathbf{B}$ is the magnetic induction, $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ are the current densities to first and second order in the electric field, f is the electron velocity distribution function, $\mathbf{v}$ is the velocity, and η is the the electron charge-to-mass ratio.

Taking the Fourier transform of (1) and (2), then combining, yields

$$\frac{c^2}{\omega_\alpha^2}[\mathbf{k}_\alpha(\mathbf{k}_\alpha . \mathbf{E}_\alpha) - k_\alpha^2 \mathbf{E}_\alpha] = -\mathbf{E}_\alpha + \frac{j}{\omega_\alpha \epsilon_0}(\mathbf{J}_\alpha^{(1)} + \mathbf{J}_\alpha^{(2)}). \tag{4}$$

The first-order current is given by

$$\mathbf{J}_\alpha^{(1)} = n_0 e \mathbf{v}_\alpha = \frac{\epsilon_0}{j\omega_\alpha}\omega_p^2 \mathbf{E}_\alpha, \tag{5}$$

where

$$\omega_p^2 = n_0 e^2/m\epsilon_0. \tag{6}$$

Equation (4) then becomes

$$\frac{c^2}{\omega_\alpha^2}[\mathbf{k}_\alpha(\mathbf{k}_\alpha . \mathbf{E}_\alpha) - k_\alpha^2 \mathbf{E}_\alpha] = -\epsilon_\alpha \mathbf{E}_\alpha + \frac{j}{\omega_\alpha \epsilon_0}\mathbf{J}_\alpha^{(2)}, \tag{7}$$

where

$$\epsilon_\alpha = 1 - \omega_p^2/\omega_\alpha^2. \tag{8}$$

Taking the dot product of (8) with respect to $\mathbf{k}_\alpha$ gives

$$\mathbf{k}_\alpha . \mathbf{E}_\alpha = \frac{j\mathbf{k}_\alpha . \mathbf{J}_\alpha^{(2)}}{\omega_\alpha \epsilon_0 \epsilon_\alpha}. \tag{9}$$

Substituting this into (7), and solving for $\mathbf{E}_\alpha$, yields

$$\mathbf{E}_\alpha = \frac{j}{\omega_\alpha \epsilon_0 \epsilon_\alpha}\frac{\mathbf{G}(\omega_\alpha, \mathbf{k}_\alpha)}{k_\alpha^2 - k_\alpha^2(\omega_\alpha)}, \tag{10}$$

where

$$\mathbf{G} = \mathbf{k}_\alpha \times (\mathbf{k}_\alpha \times \mathbf{J}_\alpha^{(2)}), \tag{11}$$

$$k_\alpha(\omega_\alpha) = \frac{\omega_\alpha}{c}\epsilon_\alpha^{\frac{1}{2}}. \tag{12}$$

3. Determination of far-field power flux density

We now determine the electric fields in the far-field zone, by taking the inverse Fourier transform, then applying essentially an asymptotic expansion technique (Lighthill 1960). The inverse transform of (10) is given by

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} \frac{j}{\omega_\alpha \epsilon_0 \epsilon_\alpha} d\mathbf{k}_\alpha \, d\omega_\alpha \exp j(\omega_\alpha t - j\mathbf{k}_\alpha . \mathbf{r}) \frac{\mathbf{G}(\omega_\alpha, \mathbf{k}_\alpha)}{k_\alpha^2(-k_\alpha^2 \omega_\alpha)}. \tag{13}$$

If we replace $\mathbf{G}$ by its spatial transform, we obtain

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} \frac{j}{\omega_\alpha \epsilon_0 \epsilon_\alpha} \frac{\exp\{-j\mathbf{k}_\alpha . (\mathbf{r} - \mathbf{r}')\}}{k_\alpha^2 - k_\alpha^2(\omega_\alpha)} \mathbf{G}(\omega_\alpha, \mathbf{r}') \exp\{j\omega_\alpha t\} \, d\mathbf{k}_\alpha \, d\omega_\alpha \, d\mathbf{r}'. \tag{14}$$

Since $\mathbf{r} \gg \mathbf{r}'$, the integral over $\mathbf{k}_\alpha$ can be evaluated in the form

$$\int_{-\infty}^{\infty} \frac{\exp\{-j\mathbf{k}_\alpha . (\mathbf{r} - \mathbf{r}')\}}{k_\alpha^2 - k_\alpha^2(\omega_\alpha)} d\mathbf{k}_\alpha = \frac{2\pi^2 \exp\{-jk_\alpha(\omega_\alpha)|\mathbf{r} - \mathbf{r}'|\}}{|\mathbf{r} - \mathbf{r}'|}$$
$$\simeq \frac{2\pi^2 \exp\{-jk_\alpha(\omega_\alpha) \mathbf{e}_r . (\mathbf{r} - \mathbf{r}')\}}{r}, \tag{15}$$

where $\mathbf{e}_r = \mathbf{r}/r$. Finally, the integration over $\mathbf{r}'$ yields

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2(2\pi)^2}\int_{-\infty}^{\infty} \frac{j}{\omega_\alpha \epsilon_0 \epsilon_\alpha} \frac{\exp\{j\omega_\alpha t - j\mathbf{k}_\alpha(\omega_\alpha) . \mathbf{r}\}}{r} \mathbf{G}[\omega_\alpha, \mathbf{k}_\alpha(\omega_\alpha)] \, d\omega_\alpha, \tag{16}$$

where $\mathbf{k}_\alpha(\omega_\alpha) = k_\alpha(\omega_\alpha)\mathbf{e}_r$. From (2), the corresponding magnetic field is given by

$$\mathbf{H}(\mathbf{r}, t) = \frac{1}{2(2\pi)^2}\int_{-\infty}^{\infty} \frac{c^2 j}{\omega_\alpha^2 \epsilon_\alpha} \frac{\exp\{j\omega_\alpha t - j\mathbf{k}_\alpha(\omega_\alpha) . \mathbf{r}\}}{r} \mathbf{k}_\alpha(\omega_\alpha) \times \mathbf{G}[\omega_\alpha, \mathbf{k}_\alpha(\omega_\alpha)] \, d\omega_\alpha. \tag{17}$$

The time-averaged power flow is given by†

$$\mathbf{P}(\mathbf{r}) = \lim_{T\to\infty} \frac{2}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} dt\,\mathrm{Re}\,\mathbf{E}(\mathbf{r},t)\times\mathbf{H}^*(\mathbf{r},t), \tag{18}$$

therefore

$$\begin{aligned}\mathbf{P}(\mathbf{r}) = \lim_{T\to 0} \frac{1}{2T(2\pi)^4}\int_{-\frac{1}{2}T}^{\frac{1}{2}T}\int_{-\infty}^{\infty} dt\,d\omega_\alpha\,d\omega'_\alpha \\ \times \mathrm{Re}\left\{\frac{c^2\exp\{j(\omega_\alpha-\omega'_\alpha)t - j[\mathbf{k}_\alpha(\omega_\alpha)-\mathbf{k}'_\alpha(\omega'_\alpha)].\mathbf{r}\}}{r^2\omega_\alpha(\omega'_\alpha)^2\epsilon_0\epsilon_\alpha\epsilon'_\alpha}\right. \\ \left.\times\,\mathbf{G}(\omega_\alpha,\mathbf{k}_\alpha(\omega_\alpha))\times[\mathbf{k}'_\alpha(\omega'_\alpha)\times\mathbf{G}(\omega'_\alpha,\mathbf{k}'_\alpha(\omega'_\alpha))]^*\right\}.\end{aligned} \tag{19}$$

If T is very large, we may take the limit

$$\int_{-\frac{1}{2}T}^{\frac{1}{2}T} \exp\{j(\omega_\alpha-\omega'_\alpha)t\}\,dt = 2\pi\delta(\omega_\alpha-\omega'_\alpha), \tag{20}$$

and (19) becomes

$$\mathbf{P}(\mathbf{r}) = \lim_{T\to\infty}\frac{1}{2T(2\pi)^3}\int_{-\infty}^{\infty} d\omega_\alpha \frac{c^2}{r^2\omega_\alpha^3\epsilon_0\epsilon_\alpha^2}\mathbf{G}(\omega_\alpha,\mathbf{k}_\alpha(\omega_\alpha))\times[\mathbf{k}_\alpha(\omega_\alpha)\times\mathbf{G}(\omega_\alpha,\mathbf{k}_\alpha(\omega_\alpha))]^*. \tag{21}$$

Upon substitution of (11) and (12), and simplification, we obtain finally

$$\mathbf{P}(\mathbf{r}) = \lim_{T\to\infty}\frac{1}{T(2\pi)^3}\int_0^{\infty}\frac{\epsilon_\alpha^{\frac{1}{2}}\omega_\alpha^2}{k_\alpha^4 r^2 c^3\epsilon_0}|\mathbf{k}_\alpha\times[\mathbf{k}_\alpha\times\mathbf{J}_\alpha^{(2)}(\omega_\alpha,\mathbf{k}_\alpha(\omega_\alpha))]|^2\,\mathbf{e}_r\,d\omega_\alpha. \tag{22}$$

An extra factor of 2 has appeared in (22), because the integration is carried out over positive frequencies only.

In what follows, it will prove more convenient to write (22) in its differential form

$$\frac{\partial^2\mathbf{P}(\mathbf{r})}{\partial\Omega\,\partial\omega_\alpha} = \lim_{TV\to\infty}\frac{\epsilon_\alpha^{\frac{1}{2}}\omega_\alpha^2 V}{(2\pi)^3 T V c^3\epsilon_0 k_\alpha^4}|\mathbf{k}_\alpha\times[\mathbf{k}_\alpha\times\mathbf{J}_\alpha^{(2)}(\omega_\alpha,\mathbf{k}_\alpha(\omega_\alpha))]|^2\,\mathbf{e}_r, \tag{23}$$

where Ω is the solid angle into which wave α is scattered and V is the scattering volume.

4. Solution of Vlasov equation

In § 3 we derived an expression for the scattered power as a function of the source current $\mathbf{J}_\alpha^{(2)}$. In § 4 we determine the latter quantity. In our derivation, we assume an isotropic unperturbed electron velocity distribution function, and the absence of a static magnetic field. The Fourier transform of (3) has the form

$$\begin{aligned}(j\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})f_\alpha + \eta(\mathbf{E}_\alpha+\mathbf{v}\times\mathbf{B}_\alpha).\frac{\partial f_{0e}}{\partial\mathbf{v}} + \frac{\eta}{(2\pi)^4}\int_{-\infty}^{\infty} d\omega_\delta\,d\omega_\epsilon\,d\mathbf{k}_\delta\,d\mathbf{k}_\epsilon(\mathbf{E}_\delta+\mathbf{v}\times\mathbf{B}_\delta) \\ \times\frac{\partial f_\epsilon}{\partial\mathbf{v}}.\delta(\mathbf{k}_\alpha-\mathbf{k}_\delta-\mathbf{k}_\epsilon)\,\delta(\omega_\alpha-\omega_\delta-\omega_\epsilon),\end{aligned} \tag{24}$$

where f_{0e} is the unperturbed electron velocity distribution, and the subscripts α, δ, and ϵ refer to waves with frequency-wavenumber pairs $(\omega_\alpha, \mathbf{k}_\alpha)$, $(\omega_\delta, \mathbf{k}_\delta)$ and

† We have ignored an additional component to the power flow given by

$$-\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} dt\,\epsilon_0\,\mathbf{E}^*.\frac{\partial\omega_\alpha\epsilon_\gamma}{\partial\mathbf{k}_\alpha}.\mathbf{E}.$$

$(\omega_\epsilon, \mathbf{k}_\epsilon)$, respectively. Since the incoming wave is plane and monochromatic, it has a spectrum of the form

$$\mathbf{E} = (2\pi)^4 \mathbf{E}_\beta \delta(\omega - \omega_\beta)\, \delta(\mathbf{k} - \mathbf{k}_\beta), \tag{25}$$

and (24) becomes

$$j(\omega_\alpha - \mathbf{k}_\alpha . \mathbf{v}) f_\alpha + \eta(\mathbf{E}_\alpha + \mathbf{v} \times \mathbf{B}_\alpha) . \frac{\partial f_{0e}}{\partial \mathbf{v}} + \sum_{\delta, \epsilon} \eta(\mathbf{E}_\delta + \mathbf{v} \times \mathbf{B}_\delta) . \frac{\partial f_\epsilon}{\partial \mathbf{v}} = 0, \tag{26}$$

where δ, ϵ in the summation run over the values

$$\delta = \beta, \quad \epsilon = \gamma, \tag{27}$$

$$\delta = \gamma, \quad \epsilon = \beta, \tag{28}$$

and γ refers to the wave for which the synchronism conditions

$$\omega_\alpha = \omega_\beta + \omega_\gamma, \quad \mathbf{k}_\alpha = \mathbf{k}_\beta + \mathbf{k}_\gamma, \tag{29}$$

hold.

We may solve (26) iteratively. The first-order solution is given by

$$f_\alpha^{(1)} = \frac{j\eta \mathbf{E}_\alpha . \partial f_{0e}/\partial \mathbf{v}}{\omega_\alpha - \mathbf{k}_\alpha . \mathbf{v}} + f_{u\alpha e}, \tag{30}$$

where $f_{u\alpha}$ is the fluctuating part of the solution (Kadomtsev 1965), which vanishes when averaged over the microscopic volume, and satisfies the equations

$$(\omega_\alpha - \mathbf{k}_\alpha . \mathbf{v}) f_{u\alpha} = 0, \tag{31}$$

$$\langle f_{u\alpha}(v) f_{u\alpha'}(\mathbf{v}')\rangle = (2\pi)^5 \delta(\mathbf{v} - \mathbf{v}')\, \delta(\omega_\alpha - \mathbf{k}_\alpha . \mathbf{v})\, \delta(\omega_\alpha - \omega_{\alpha'})\, \delta(\mathbf{k}_\alpha - \mathbf{k}_{\alpha'}) f_0(v). \tag{32}$$

Substituting in (26), we obtain the second-order solution

$$f_\alpha^{(2)} = \frac{j\eta}{\omega_\alpha - \mathbf{k}_\alpha . \mathbf{v}} \sum_{\delta, \epsilon} (\mathbf{E}_\delta + \mathbf{v} \times \mathbf{B}_\delta) . \frac{\partial}{\partial \mathbf{v}} \left(\frac{j\eta \mathbf{E}_\epsilon \partial f_{0e}/\partial \mathbf{v}}{\omega_\epsilon - \mathbf{k}_\epsilon . \mathbf{v}} \right) + \frac{j\eta}{\omega_\alpha - \mathbf{k}_\alpha . \mathbf{v}} \sum_{\delta, \epsilon} (\mathbf{E}_\delta + \mathbf{v} \times \mathbf{B}_\delta) . \frac{\partial f_{u\epsilon\alpha}}{\partial \mathbf{v}}. \tag{33}$$

5. Second-order source currents

The first-order currents, obtained by substitution of (30) into

$$\mathbf{J}_\alpha^{(1)} = e \int_{-\infty}^{\infty} f_\alpha^{(1)} \mathbf{v}\, dv, \tag{34}$$

need not be considered further, since the contribution from the first term on the right-hand side of (30) has already been accounted for by (5); the second term does not contain $\mathbf{E}_\beta$ as a factor, and therefore does not contribute to the scattering.

Substituting (33) into

$$\mathbf{J}_\alpha^{(2)} = e \int f_\alpha^{(2)} \mathbf{v}\, d\mathbf{v} \tag{35}$$

gives a second-order current

$$\mathbf{J}_\alpha^{(2)} = \mathbf{J}_{s\alpha}^{(2)} + \mathbf{J}_{u\alpha}^{(2)}, \tag{36}$$

where

$$\mathbf{J}_{s\alpha}^{(2)} = -\sum_{\delta,\,\epsilon}\int_{-\infty}^{\infty}\frac{\eta^2 e}{\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v}}\mathbf{v}\,d\mathbf{v}(\mathbf{E}_\delta+\mathbf{v}\times\mathbf{B}_\delta).\frac{\partial}{\partial\mathbf{v}}\left(\frac{\mathbf{E}_\epsilon.\partial f_{0e}/\partial\mathbf{v}}{\omega_\epsilon-\mathbf{k}_\epsilon.\mathbf{v}}\right), \tag{37}$$

$$\mathbf{J}_{u\alpha}^{(2)} = \sum_{\delta,\,\epsilon}\int_{-\infty}^{\infty}\frac{j\eta e}{\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v}}\mathbf{v}\,d\mathbf{v}(\mathbf{E}_\delta+\mathbf{v}\times\mathbf{B}_\delta).\frac{\partial f_{ue\epsilon}}{\partial\mathbf{v}}. \tag{38}$$

6. Source current from collective effects

We shall concentrate first on evaluating $\mathbf{J}_{s\alpha}^{(2)}$, the source current due to collective effects. A partial integration reduces this to

$$\mathbf{J}_{s\alpha}^{(2)} = \sum_{\delta,\,\epsilon}\eta^2 e\int_{-\infty}^{\infty}\left(\frac{\mathbf{E}_\epsilon.\partial f_{0e}/\partial\mathbf{v}}{\omega_\epsilon-\mathbf{k}_\epsilon.\mathbf{v}}\right)\left\{\frac{(\mathbf{E}_\delta+\mathbf{v}\times\mathbf{B}_\delta).\mathbf{k}_\alpha}{(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})^2}\mathbf{v}+\frac{\mathbf{E}_\delta+\mathbf{v}\times\mathbf{B}_\delta}{\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v}}\right\}d\mathbf{v}. \tag{39}$$

A second partial integration, followed by expansion of the summation according to (27) and (28), yields

$$\begin{aligned}\mathbf{J}_{s\alpha}^{(2)} = -\eta^2 e\int_{-\infty}^{\infty} f_{0e}\,d\mathbf{v}\Bigg\{&\frac{\mathbf{E}_\gamma}{\omega_\gamma-\mathbf{k}_\gamma.\mathbf{v}}\,\frac{(\mathbf{E}_\beta+\mathbf{v}\times\mathbf{B}_\beta).\mathbf{k}_\alpha}{(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})^2}\\ &+\frac{2\mathbf{E}_\gamma.\mathbf{k}_\alpha}{\omega_\gamma-\mathbf{k}_\gamma.\mathbf{v}}\,\frac{(\mathbf{E}_\beta+\mathbf{v}\times\mathbf{B}_\beta).\mathbf{k}_\alpha}{(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})^3}\mathbf{v}+\frac{\mathbf{E}_\gamma\times\mathbf{B}_\beta.\mathbf{k}_\alpha}{(\omega_\gamma-\mathbf{k}_\gamma.\mathbf{v})\,(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})^2}\mathbf{v}\\ &+\frac{(E_\gamma.\mathbf{k}_\alpha)(\mathbf{E}_\beta+\mathbf{v}\times\mathbf{B}_\beta)}{(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})^2(\omega_\gamma-\mathbf{k}_\gamma.\mathbf{v})}+\frac{\mathbf{E}_\gamma\times\mathbf{B}_\beta}{(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})\,(\omega_\gamma-\mathbf{k}_\gamma.\mathbf{v})}\\ &+\frac{\mathbf{k}_\gamma.\mathbf{E}_\gamma}{(\omega_\gamma-\mathbf{k}_\gamma.\mathbf{v})^2}\left[\frac{(\mathbf{E}_\beta+\mathbf{v}\times\mathbf{B}_\beta).\mathbf{k}_\alpha}{(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})^2}\mathbf{v}+\frac{\mathbf{E}_\beta+\mathbf{v}\times\mathbf{B}_\beta}{\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v}}\right]\\ &+\frac{\mathbf{E}_\beta}{\omega_\beta-\mathbf{k}_\beta.\mathbf{v}}\,\frac{\mathbf{E}_\gamma.\mathbf{k}_\alpha}{(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})^2}+\frac{(2\mathbf{k}_\alpha.\mathbf{E}_\beta)\,(\mathbf{k}_\alpha.\mathbf{E}_\gamma)}{(\omega_\beta-\mathbf{k}_\beta.\mathbf{v})\,(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})^3}\mathbf{v}\\ &+\frac{(\mathbf{k}_\alpha.\mathbf{E}_\beta)\,\mathbf{E}_\gamma}{(\omega_\beta-\mathbf{k}_\beta.\mathbf{v})\,(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})^2}\Bigg\}.\end{aligned} \tag{40}$$

In obtaining (40), we have used the relations

$$\mathbf{k}_\alpha.\mathbf{E}_\alpha = 0, \quad \mathbf{k}_\beta.\mathbf{E}_\beta = 0, \quad \mathbf{B}_\gamma = 0, \tag{41}$$

which follow from the transverse and longitudinal character of the linearized electromagnetic and electrostatic waves, respectively.

We shall find it more convenient to write (40) in the form

$$\mathbf{k}_\alpha\times\mathbf{k}_\alpha\times\mathbf{J}_{s\alpha}^{(2)} = -\frac{\eta^2 e E_\beta E_\gamma k_\gamma}{\omega_\alpha}\int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,dv\,\mathbf{V}_s(\mathbf{v}), \tag{42}$$

where

$$\mathbf{V}_s(\mathbf{v}) = \mathbf{k}_\alpha \times \mathbf{k}_\alpha \times \frac{\omega_\alpha}{k_\gamma^2}\left\{\frac{\mathbf{k}_\gamma[(\omega_\beta - \mathbf{k}_\beta.\mathbf{v})(\mathbf{e}_\beta.\mathbf{k}_\alpha) + (\mathbf{k}_\alpha.\mathbf{k}_\beta)(\mathbf{e}_\beta.\mathbf{v})]}{\omega_\beta(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^2(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})}\right.$$
$$+2\frac{(\mathbf{k}_\alpha.\mathbf{k}_\gamma)[(\omega_\beta - \mathbf{k}_\beta.\mathbf{v})(e_\beta.\mathbf{k}_\alpha) + (\mathbf{k}_\alpha.\mathbf{k}_\beta)(\mathbf{e}_\beta.\mathbf{v})]}{\omega_\beta(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^3(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})}\mathbf{v}$$
$$+\frac{[(\mathbf{k}_\alpha.\mathbf{k}_\beta)(\mathbf{e}_\beta.\mathbf{k}_\alpha) - (\mathbf{e}_\beta.\mathbf{k}_\alpha)(\mathbf{k}_\beta.\mathbf{k}_\gamma)]}{(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^2\omega_\beta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})}\mathbf{v}$$
$$+\frac{(\mathbf{k}_\alpha.\mathbf{k}_\gamma)[(\omega_\beta - \mathbf{k}_\beta.\mathbf{v})\mathbf{e}_\beta + \mathbf{k}_\beta(\mathbf{e}_\beta.\mathbf{v})]}{(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^2\omega_\beta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})} + \frac{[\mathbf{k}_\beta(\mathbf{e}_\beta.\mathbf{k}_\alpha) - \mathbf{e}_\beta(\mathbf{k}_\beta.\mathbf{k}_\gamma)]}{\omega_\beta(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})(\omega_\gamma.\mathbf{k}_\gamma.\mathbf{v})}$$
$$+\frac{k_\gamma^2}{(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})^2}\left[\frac{(\omega_\beta - \mathbf{k}_\beta.\mathbf{v})(\mathbf{e}_\beta.\mathbf{k}_\alpha) + (\mathbf{k}_\alpha.\mathbf{k}_\beta)(\mathbf{e}_\beta.\mathbf{v})}{\omega_\beta(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^2}\mathbf{v}\right.$$
$$\left.+\frac{(\omega_\beta - \mathbf{k}_\beta.\mathbf{v})\mathbf{e}_\beta + (\mathbf{e}_\beta.\mathbf{v})\mathbf{k}_\beta}{\omega_\beta(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})}\right] + \frac{\mathbf{e}_\beta}{\omega_\beta - \mathbf{k}_\beta.\mathbf{v}}\frac{(\mathbf{k}_\alpha.\mathbf{k}_\gamma)}{(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^2}$$
$$\left.+\frac{2(\mathbf{e}_\beta.\mathbf{k}_\alpha)(\mathbf{k}_\alpha.\mathbf{k}_\gamma)\mathbf{v}}{(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^3(\omega_\beta - \mathbf{k}_\beta.\mathbf{v})} + \frac{(\mathbf{e}_\beta.\mathbf{k}_\alpha)\mathbf{k}_\gamma}{(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^2(\omega_\beta - \mathbf{k}_\beta.\mathbf{v})}\right\}. \tag{43}$$

We know from the synchronism conditions (29) that

$$(\omega_\beta - \mathbf{k}_\beta.\mathbf{v}) = (\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v}) - (\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v}), \tag{44}$$

$$\mathbf{k}_\alpha \times \mathbf{k}_\alpha \times \mathbf{k}_\alpha = \mathbf{k}_\alpha \times \mathbf{k}_\alpha \times (\mathbf{k}_\beta + \mathbf{k}_\gamma) = 0. \tag{45}$$

Substituting these into (43), and collecting terms, yields

$$\mathbf{V}_s(\mathbf{v}) = \mathbf{k}_\alpha \times \mathbf{k}_\alpha \times \frac{\omega_\alpha}{\omega_\beta k_\gamma^2}\left\{\frac{1}{(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})}\left[\frac{[\mathbf{k}_\alpha.(\mathbf{k}_\beta - \mathbf{k}_\gamma)](\mathbf{e}_\beta.\mathbf{v})\mathbf{k}_\gamma}{(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^2}\right.\right.$$
$$\left.+\frac{k_\alpha^2(\mathbf{e}_\beta.\mathbf{k}_\alpha)\mathbf{v}}{(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^2} + 2\frac{(\mathbf{k}_\alpha.\mathbf{k}_\gamma)(\mathbf{k}_\alpha.\mathbf{k}_\beta)(\mathbf{e}_\beta.\mathbf{v})\mathbf{v}}{(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^3}\right] + \frac{k_\gamma^2}{(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})^2}\left[\frac{(\mathbf{e}_\beta.\mathbf{k}_\alpha)\mathbf{v}}{\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v}}\right.$$
$$\left.+\frac{(\mathbf{k}_\alpha.\mathbf{k}_\beta)(\mathbf{e}_\beta.\mathbf{v})\mathbf{v}}{(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^2} + \mathbf{e}_\beta - \frac{(\mathbf{e}_\beta.\mathbf{v})\mathbf{k}_\gamma}{\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v}}\right] + \frac{(\mathbf{k}_\alpha.\mathbf{k}_\gamma)(\mathbf{k}_\beta.\mathbf{v})\mathbf{e}_\beta}{(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^2(\omega_\beta - \mathbf{k}_\beta.\mathbf{v})}$$
$$\left.+2\frac{(\mathbf{e}_\beta.\mathbf{k}_\gamma)(\mathbf{k}_\alpha.\mathbf{k}_\gamma)(\mathbf{k}_\beta.\mathbf{v})\mathbf{v}}{(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^2(\omega_\beta - \mathbf{k}_\beta.\mathbf{v})} + \frac{(\mathbf{e}_\beta.\mathbf{k}_\gamma)(\mathbf{k}_\beta.\mathbf{v})\mathbf{k}_\gamma}{(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})^2(\omega_\beta - \mathbf{k}_\beta.\mathbf{v})}\right\}. \tag{46}$$

7. Source current from discrete particle effects

The source current due to discrete particle effects is obtained in the same manner as the source current from collective effects. We expand the summation in (38), using only the term corresponding to (27); this is the only term dependent on the incoming electromagnetic wave, and therefore represents scattering. We obtain

$$\mathbf{J}_{u\alpha}^{(2)} = \int_{-\infty}^{\infty} \frac{j\eta e}{\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v}}\mathbf{v}\,d\mathbf{v}(\mathbf{E}_\beta + \mathbf{v}\times\mathbf{B}_\beta).\frac{\partial f_{u\gamma e}}{\partial \mathbf{v}}. \tag{47}$$

A partial integration reduces this to the form

$$\mathbf{J}_{u\alpha}^{(2)} = -j\eta e\int_{-\infty}^{\infty} f_{u\gamma e}\,d\mathbf{v}\left[\frac{\mathbf{E}_\beta+\mathbf{v}\times\mathbf{B}_\beta}{\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v}}+\frac{(\mathbf{E}_\beta+\mathbf{v}\times\mathbf{B}_\beta).\mathbf{k}_\alpha}{(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})^2}\mathbf{v}\right]. \tag{48}$$

Here again we find it more useful to write this as

$$\mathbf{k}_\alpha\times\mathbf{k}_\alpha\times\mathbf{J}_{u\alpha}^{(2)} = -\frac{j\eta e E_\beta}{\omega_\alpha}\int_{-\infty}^{\infty} f_{u\gamma e}\,d\mathbf{v}\,\mathbf{V}_u(\mathbf{v}), \tag{49}$$

where

$$\mathbf{V}_u(\mathbf{v}) = \mathbf{k}_\alpha\times\left\{\mathbf{k}_\alpha\times\omega_\alpha\left[\frac{(\omega_\beta-\mathbf{k}_\beta.\mathbf{v})\,\mathbf{e}_\beta+\mathbf{k}_\beta(\mathbf{e}_\beta.\mathbf{v})}{\omega_\beta(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})}\right.\right.$$
$$\left.\left.+\frac{(\omega_\beta-\mathbf{k}_\beta.\mathbf{v})\,(\mathbf{e}_\beta.\mathbf{k}_\alpha)+(\mathbf{k}_\alpha.\mathbf{k}_\beta)\,(\mathbf{e}_\beta.\mathbf{v})}{\omega_\beta(\omega_\alpha-\mathbf{k}_\alpha.\mathbf{v})^2}\mathbf{v}\right]\right\}. \tag{50}$$

8. Scattering formula

We are now in a position to obtain the final scattering formula. Substituting (42) and (49) into (23) gives

$$\frac{\partial^2 P}{\partial\Omega\,\partial\omega_\alpha} = \lim_{TV\to\infty}\frac{\epsilon_\alpha^{\frac{1}{2}}\eta^2 e^2|E_\beta|^2\,V}{(2\pi)^3\,T\,V c^3\epsilon_0\,k_\alpha^4}\left|\left\{\eta E_\gamma\,k_\gamma\int_{-\infty}^{\infty} f_{0e}(\mathbf{v})d\mathbf{v}\,V_s(\mathbf{v})+j\int_{-\infty}^{\infty} f_{u\gamma e}\,d\mathbf{v}\,\mathbf{V}_u(\mathbf{v})\right\}\right|^2. \tag{51}$$

Since the incoming flux is given by

$$S_\beta = 2\epsilon_0\,\epsilon_\beta^{\frac{1}{2}}c|E_\beta|^2, \tag{52}$$

and the classical electron radius by

$$r_0 = e^2/(4\pi\epsilon_0 mc^2), \tag{53}$$

the scattering formula can be written simply as

$$\frac{\partial^2 P}{\partial\Omega\,\partial\omega_\alpha} = \lim_{TV\to\infty}\frac{r_0^2 S_\beta\,V}{\pi T\,V k_\alpha^4}\left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{\frac{1}{2}}\left|\eta E_\gamma\,k_\gamma\int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,d\mathbf{v}\,V_s(\mathbf{v})+j\int_{-\infty}^{\infty} f_{u\gamma e}\,d\mathbf{v}\,\mathbf{V}_u(\mathbf{v})\right|^2. \tag{54}$$

Expanding the squared term gives

$$\frac{\partial^2 P}{\partial\Omega\,\partial\omega_\alpha} = \lim_{TV-\infty}\frac{r_0^2 S_\beta V}{\pi T\,V k_\alpha^4}\left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{\frac{1}{2}}\left\{\eta^2 k_\gamma^2[E_\gamma E_\gamma^*]\left|\int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,\mathbf{V}_s(\mathbf{v})\,d\mathbf{v}\right|^2\right.$$
$$-2\,\mathrm{Re}\,j\eta k_\gamma\left(\int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,d\mathbf{v}\,V_s(\mathbf{v})\right).\left(\int_{-\infty}^{\infty}[f_{u\gamma e}(\mathbf{v})^* E_\gamma]\,\mathbf{V}_u^*(\mathbf{v})\,d\mathbf{v}\right)$$
$$\left.+\int_{-\infty}^{\infty} d\mathbf{v}\,d\mathbf{v}'[f_{u\gamma e}(\mathbf{v})f_{u\gamma e}^*(\mathbf{v}')]\,\mathbf{V}_u(\mathbf{v}).\mathbf{V}_u^*(\mathbf{v}')\right\}. \tag{55}$$

This equation is the general scattering formula we have sought to derive. If one knows the spectrum corresponding to $|E_\gamma|^2$, $f_{u\gamma e}^* E_\gamma$ and $|f_{u\gamma e}^*|^2$, and the unperturbed velocity distributions then the scattered power is determined.

10. Incoherent scatter

We shall now take up the case of incoherent scatter, where it is possible to evaluate (55) explicitly. In this case the assumption that the charged particle motions are random allows one to evaluate the products of the fluctuating quantities in the equation. In the appendix we show that these products are given by

$$\lim_{TV\to\infty} \frac{1}{TV} f_{u\gamma e}(\mathbf{v}) f^*_{u\gamma e}(\mathbf{v}') = 2\pi\delta(\mathbf{v}-\mathbf{v}')\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v}) f_{0e}(\mathbf{v}), \tag{56}$$

$$\lim_{TV\to\infty} \frac{1}{TV} f^*_{u\gamma e}(\mathbf{v})\, E_\gamma = \frac{2\pi e j}{\epsilon_0 \epsilon_\gamma k_\gamma} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v}), \tag{57}$$

$$\lim_{TV\to\infty} \frac{1}{TV} |E_\gamma|^2 = \frac{2\pi e^2}{\epsilon_0^2 |\epsilon_\gamma|^2 k_\gamma^2} \left[\int_{-\infty}^{\infty} d\mathbf{v} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v}) + \int_{-\infty}^{\infty} d\mathbf{v} f_{0i}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})\right]. \tag{58}$$

Substituting these expressions, and integrating over $\mathbf{v}'$, reduces (55) to the form

$$\frac{\partial^2 P}{\partial\Omega\,\partial\omega_\alpha} = 2r_0^2 S_\beta V \left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{\frac{1}{2}} \frac{1}{k_\alpha^4} \left\{(\mathbf{L}_1.\mathbf{L}_1^*)\left[\int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})\,d\mathbf{v} + \int_{-\infty}^{\infty} f_{0i}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})\,d\mathbf{v}\right] + 2\,\mathrm{Re}\,\mathbf{L}_1.\mathbf{L}_2 + \mathbf{L}_3\right\}, \tag{59}$$

where
$$\mathbf{L}_1 = \frac{\eta e}{\epsilon_0 \epsilon_\gamma} \int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,\mathbf{V}_s(\mathbf{v})\,d\mathbf{v}, \tag{60}$$

$$\mathbf{L}_2 = \int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})\,\mathbf{V}_u^*(\mathbf{v})\,d\mathbf{v}, \tag{61}$$

$$\mathbf{L}_3 = \int_{-\infty}^{\infty} d\mathbf{v} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})\,\mathbf{V}_u(\mathbf{v}).\mathbf{V}_u(\mathbf{v})^*. \tag{62}$$

Because of the delta function and (44), we can replace the factor $(\omega_\alpha - \mathbf{k}_\alpha.\mathbf{v})$ by $(\omega_\beta - \mathbf{k}_\beta.\mathbf{v})$ in the definitions for $\mathbf{L}_2$ and $\mathbf{L}_3$. Carrying this out, along with the application of (45), yields the simpler equations

$$\mathbf{L}_2 = \int_{-\infty}^{\infty} d\mathbf{v} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})\,\mathbf{W}_u(\mathbf{v}), \tag{63}$$

$$\mathbf{L}_3 = \int_{-\infty}^{\infty} d\mathbf{v} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})\,\mathbf{W}_u(\mathbf{v}).\mathbf{W}_u^*(\mathbf{v}), \tag{64}$$

where
$$\mathbf{W}_u = \frac{\omega_\alpha}{\omega_\beta} \mathbf{k}_\alpha \times \left\{\mathbf{k}_\alpha \times \left[\mathbf{e}_\beta + \frac{(\mathbf{e}_\beta.\mathbf{k}_\alpha)\,\mathbf{v} - (\mathbf{e}_\beta.\mathbf{v})\,\mathbf{k}_\gamma}{\omega_\beta - \mathbf{k}_\beta.\mathbf{v}} + \frac{(\mathbf{k}_\alpha.\mathbf{k}_\beta)\,(\mathbf{e}_\beta.\mathbf{v})\,\mathbf{v}}{(\omega_\beta - \mathbf{k}_\beta.\mathbf{v})^2}\right]\right\}. \tag{65}$$

Equation (59), along with the definition given by (46), (60), (63)–(65), is our final result describing scattering by random density fluctuations.

11. High-frequency expansion for incoherent scatter

Equation (59) is a final result in the sense that it specifies completely the scattered power once the unperturbed velocity distribution functions are known. It will be useful, however, to expand this equation in powers of $1/\omega_\beta$, in order to interpret the meaning of the result, and to compare with previous work.

Let us first expand $\mathbf{L}_1$ in (60) to second order in ω_β^{-1}. This yields, upon application of (44), the relation

$$\mathbf{V}_s(\mathbf{v}) = \mathbf{k}_\alpha \times \left\{\mathbf{k}_\alpha \times \frac{\omega_\alpha}{\omega_\beta k_\gamma^2}\left\{\frac{[\mathbf{k}_\alpha.(\mathbf{k}_\beta - \mathbf{k}_\gamma)](\mathbf{e}_\beta.\mathbf{v})\mathbf{k}_\gamma}{\omega_\beta^2(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})} + \frac{k_\alpha^2(\mathbf{e}_\beta.\mathbf{k}_\alpha)\mathbf{v}}{\omega_\beta^2(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})} + \frac{k_\gamma^2}{(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})^2}\right.\right.$$
$$\left.\left.\times\left[\left(\frac{(\mathbf{e}_\beta.\mathbf{k}_\alpha)\mathbf{v} - (\mathbf{e}_\beta.\mathbf{v})\mathbf{k}_\gamma}{\omega_\beta}\right)\left(1 + \frac{\mathbf{k}_\beta.\mathbf{v}}{\omega_\beta} - \frac{\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v}}{\omega_\beta}\right) + \mathbf{e}_\beta + \frac{(\mathbf{k}_\alpha.\mathbf{k}_\beta)(\mathbf{e}_\beta.\mathbf{v})\mathbf{v}}{\omega_\beta^2}\right]\right\}\right\}. \tag{66}$$

If we substitute into this equation the vector identity

$$\mathbf{v} = k_\gamma^{-2}[\mathbf{k}_\gamma(\mathbf{k}_\gamma.\mathbf{v}) - \mathbf{k}_\gamma \times \mathbf{k}_\gamma \times \mathbf{v}], \tag{67}$$

and collect terms, we obtain

$$\mathbf{V}_s(\mathbf{v}) = \frac{\omega_\alpha}{\omega_\beta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})^2}\mathbf{k}_\alpha \times \left\{\mathbf{k}_\alpha \times \left\{\mathbf{e}_\beta + \frac{(\mathbf{k}_\alpha.\mathbf{k}_\beta)(\mathbf{e}_\beta.\mathbf{k}_\alpha)}{k_\gamma^4\omega_\beta^2}[\omega_\gamma^2 - (\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})^2]\mathbf{k}_\gamma\right\}\right\}$$
$$+ \text{ terms in } (\mathbf{k}_\gamma \times \mathbf{k}_\gamma \times \mathbf{v}). \tag{68}$$

Finally, substituting into (60) gives

$$\mathbf{L}_1 = -\frac{\omega_\alpha}{\omega_\beta \epsilon_\gamma}\mathbf{k}_\alpha \times \left\{\mathbf{k}_\alpha\left\{\chi_{e\gamma}\left[\mathbf{e}_\beta + \frac{(\mathbf{k}_\alpha.\mathbf{k}_\beta)(\mathbf{e}_\beta.\mathbf{k}_\alpha)}{k_\gamma^4}\left(\frac{\omega_\gamma}{\omega_\beta}\right)^2\mathbf{k}_\gamma\right]\right.\right.$$
$$\left.\left. + \frac{(\mathbf{k}_\alpha.\mathbf{k}_\beta)(\mathbf{e}_\beta.\mathbf{k}_\alpha)}{k_\gamma^4}\left(\frac{\omega_p}{\omega_\beta}\right)^2\mathbf{k}_\gamma\right\}\right\}, \tag{69}$$

where

$$\chi = -\frac{\eta e}{\epsilon_0}\int_{-\infty}^{\infty}\frac{f_0(\mathbf{v})\,d\mathbf{v}}{(\omega - \mathbf{k}.\mathbf{v})^2} \tag{70}$$

is the charged particle susceptibility. In obtaining this result, we have ignored the terms in $\mathbf{V}_s(\mathbf{v})$ containing $(\mathbf{k}_\gamma \times \mathbf{k}_\gamma \times \mathbf{v})$, since these give rise to terms of order $(v_e/c)^2$, and higher, when dealing with isotropic electron velocity distribution functions.

Let us now expand $\mathbf{L}_2$ in (63). To second order in ω_β^{-1} we obtain

$$\mathbf{L}_2 = \int d\mathbf{v} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})\frac{\omega_\alpha}{\omega_\beta}\mathbf{k}_\alpha \times \left\{\mathbf{k}_\alpha \times \left\{\mathbf{e}_\beta + [(\mathbf{e}_\beta.\mathbf{k}_\alpha)\mathbf{v} - (\mathbf{e}_\beta.\mathbf{v})\mathbf{k}_\gamma]\right.\right.$$
$$\left.\left.\times\left[\frac{1}{\omega_\beta} + \frac{\mathbf{k}_\beta.\mathbf{v}}{\omega_\beta^2}\right] + \frac{(\mathbf{k}_\alpha.\mathbf{k}_\beta)(\mathbf{e}_\beta.\mathbf{v})\mathbf{v}}{\omega_\beta^2}\right\}\right\}. \tag{71}$$

We again expand $\mathbf{v}$ according to (67), and ignore the component perpendicular to $\mathbf{k}_\gamma$. This yields, after replacing $(\mathbf{k}_\gamma.\mathbf{v})$ by ω_γ, the result

$$\mathbf{L}_2 = \left(\frac{\omega_\alpha}{\omega_\beta}\right)\mathbf{k}_\alpha \times \left\{\mathbf{k}_\alpha \times \left[\mathbf{e}_\beta + \left(\frac{\omega_\gamma}{\omega_\beta}\right)^2\frac{(\mathbf{e}_\beta.\mathbf{k}_\alpha)(\mathbf{k}_\alpha.\mathbf{k}_\beta)\mathbf{k}_\gamma}{k_\gamma^4}\right]\right\}\int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})\,d\mathbf{v}. \tag{72}$$

A similar procedure for $\mathbf{L}_3$ yields the expression

$$\mathbf{L}_3 = \left(\frac{\omega_\alpha}{\omega_\beta}\right)^2 (\mathbf{k}_\alpha \times (\mathbf{k}_\alpha \times \mathbf{e}_\beta)) . \left\{\mathbf{k}_\alpha \times \left[\mathbf{k}_\alpha \times \left(\mathbf{e}_\beta + 2\left(\frac{\omega_\gamma}{\omega_\beta}\right)^2 \frac{(\mathbf{e}_\beta . \mathbf{k}_\alpha)(\mathbf{k}_\alpha . \mathbf{k}_\beta)\mathbf{k}_\gamma}{k_\gamma^4}\right)\right]\right\} \times \int_{-\infty}^{\infty} f_{0e}(\mathbf{v})(\omega_\gamma - \mathbf{k}_\gamma . \mathbf{v})\, d\mathbf{v}. \quad (73)$$

We now substitute the expressions for $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$ into (59), to obtain

$$\begin{aligned}\frac{\partial^2 P}{\partial\Omega\,\partial\omega_\alpha} &= r_0^2 S_\beta V \left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{\frac{1}{2}} \frac{2\omega_\alpha^2}{|\epsilon_\gamma|^2 k_\alpha^4 \omega_\beta^2} (\mathbf{k}_\alpha \times (\mathbf{k}_\alpha \times \mathbf{e}_\beta)) \\ &\quad . \left\{\left[\mathbf{k}_\alpha \times \left[\mathbf{k}_\alpha \times \left(\mathbf{e}_\beta + 2\frac{(\mathbf{k}_\alpha . \mathbf{k}_\beta)(\mathbf{e}_\beta . \mathbf{k}_\alpha)}{k_\gamma^4} \frac{\omega_\gamma^2}{\omega_\beta^2}\mathbf{k}_\gamma\right)\right]\right]\right. \\ &\quad \times \left[(|\chi_{e\gamma}|^2 - 2\,\mathrm{Re}\,\chi_{e\gamma}\epsilon_\gamma^* + |\epsilon_\gamma|^2)\int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma . \mathbf{v})\, d\mathbf{v}\right. \\ &\quad \left. + |\chi_{e\gamma}|^2 \int f_{0i}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma . \mathbf{v})\, d\mathbf{v}\right] + 2(\mathbf{k}_\alpha \times (\mathbf{k}_\alpha \times \mathbf{k}_\gamma)) \\ &\quad \times \frac{(\mathbf{k}_\alpha . \mathbf{k}_\beta)(\mathbf{e}_\beta . \mathbf{k}_\alpha)}{k_\gamma^4} \frac{\omega_p^2}{\omega_\beta^2}\left[\mathrm{Re}\,\chi_{e\gamma}\int_{-\infty}^{\infty} f_{0i}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma . \mathbf{v})\, d\mathbf{v}\right. \\ &\quad \left.\left. + \mathrm{Re}\,(\chi_{e\gamma} - \epsilon_\gamma)\int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma . \mathbf{v})\, d\mathbf{v}\right]\right\}. \end{aligned} \quad (74)$$

Using the identities

$$\epsilon_\gamma = 1 + \chi_{e\gamma} + \chi_{i\gamma}, \quad (75)$$

$$|\epsilon_\gamma|^2 - 2\,\mathrm{Re}\,\epsilon_\gamma \chi_{e\gamma}^* + |\chi_{e\gamma}|^2 = |1 + \chi_{i\gamma}|^2, \quad (76)$$

and rearranging, gives the final form for the expanded incoherent scattering formula as

$$\begin{aligned}\frac{\partial^2 P}{\partial\Omega\,\partial\omega_\alpha} &= \frac{2r_0^2 S_\beta V}{|\epsilon_\gamma^2|}\left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{\frac{1}{2}}\left(\frac{\omega_\alpha}{\omega_\beta}\right)^2 \left\{\left[|1 + \chi_{i\gamma}|^2 \int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma . \mathbf{v})\, d\mathbf{v}\right.\right. \\ &\quad \left. + |\chi_{e\gamma}|^2 \int_{-\infty}^{\infty} f_{0i}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma . \mathbf{v})\, d\mathbf{v}\right]\left[1 - \frac{(\mathbf{e}_\beta . \mathbf{k}_\alpha)^2}{k_\alpha^2}\right. \\ &\quad \left. + 2\frac{(\mathbf{e}_\beta . \mathbf{k}_\alpha)^2(\mathbf{k}_\alpha . \mathbf{k}_\beta)^2}{k_\alpha^2 k_\gamma^4}\frac{\omega_\gamma^2}{\omega_\beta^2}\right] + 2\left[\mathrm{Re}\,\chi_{e\gamma}\int_{-\infty}^{\infty} f_{0i}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma . \mathbf{v})\, d\mathbf{v}\right. \\ &\quad \left.\left. - \mathrm{Re}\,(1 + \chi_{i\gamma})\int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma . \mathbf{v})\, d\mathbf{v}\right]\frac{(\mathbf{e}_\beta . \mathbf{k}_\alpha)^2(\mathbf{k}_\alpha . \mathbf{k}_\beta)^2}{k_\alpha^2 k_\gamma^4}\frac{\omega_p^2}{\omega_\beta^2}\right\}. \end{aligned} \quad (77)$$

When the frequency of the incident electromagnetic wave ω_β becomes very large compared with ω_γ and ω_p, the high-frequency incoherent scattering formula (Bekefi 1966)

$$\begin{aligned}\frac{\partial^2 P}{\partial\Omega\,\partial\omega_\alpha} &= \frac{2r_0^2 S_\beta V}{|\epsilon_\alpha|^2}\left[|1 + \chi_{i\gamma}|^2 \int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma . \mathbf{v})\, d\mathbf{v}\right. \\ &\quad \left. + |\chi_{e\gamma}|^2 \int_{-\infty}^{\infty} f_{0i}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma . \mathbf{v})\, d\mathbf{v}\right]\left[\frac{1 - (\mathbf{e}_\beta . \mathbf{k}_\alpha)^2}{k_\alpha^2}\right] \end{aligned} \quad (78)$$

is retrieved. In the case of backscatter, where $(\mathbf{e}_\beta.\mathbf{k}_\alpha)=0$, (77) reduces to the even simpler form

$$\frac{\partial^2 P}{\partial\Omega\,\partial\omega_\alpha}=\frac{2r_0^2 S_\beta V}{|\epsilon_\alpha|^2}\left[|1+\chi_{i\gamma}|^2\int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma-\mathbf{k}_\gamma.\mathbf{v})\,d\mathbf{v} + |\chi_{e\gamma}|^2\int_{-\infty}^{\infty} f_{0i}(\mathbf{v})\,\delta(\omega_\gamma-\mathbf{k}_\gamma.\mathbf{v})\,d\mathbf{v}\right]. \quad (79)$$

12. Scattering in case of strongly driven plasma waves

In the case where the coherent waves are so strongly driven by an external source that the random fluctuations of the charged particles can be ignored, we may ignore the terms involving $f_{u\gamma e}$ in (55), and the scattering is then given simply by

$$\frac{\partial^2 P}{\partial\Omega\,\partial\omega_\alpha}=\frac{r_0^2 S_\beta V}{\pi k_\alpha^4}\left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{\frac{1}{2}}\eta^2 k_\gamma^2\left|\int_{-\infty}^{\infty} f_{0e}(\mathbf{v})\,\mathbf{V}_s(\mathbf{v})\,d\mathbf{v}\right|^2 \lim_{TV\to\infty}\frac{|E_\gamma|^2}{TV}. \quad (80)$$

As in the case of incoherent scatter, it is useful to determine the behaviour for a high-frequency incident wave. Substituting (60) and (69) into this formula reduces it to the form

$$\frac{\partial^2 P}{\partial\Omega\,\partial\omega_\alpha}=\frac{r_0^2 S_\beta V}{\pi}\left(\frac{\epsilon_\alpha}{\epsilon_\beta}\right)^{\frac{1}{2}}\frac{k_\gamma^2\epsilon_0^2}{e^2}\left(\frac{\omega_\alpha}{\omega_\beta}\right)^2\left\{|\chi_{e\gamma}|^2\left[1-\frac{(\mathbf{e}_\beta.\mathbf{k}_\alpha)^2}{k_\alpha^2}+2\frac{(\mathbf{e}_\beta.\mathbf{k}_\alpha)^2(\mathbf{k}_\alpha.\mathbf{k}_\beta)^2}{k_\alpha^2\,k_\gamma^4}\left(\frac{\omega_\gamma}{\omega_\beta}\right)^2\right] + 2\,\mathrm{Re}\,\chi_{e\gamma}\left[\frac{(\mathbf{e}_\beta.\mathbf{k}_\alpha)^2(\mathbf{k}_\alpha.\mathbf{k}_\beta)^2}{k_\alpha^2 k_\gamma^4}\left(\frac{\omega_p}{\omega_\beta}\right)^2\right]\right\}\lim_{TV\to\infty}\frac{|E_\gamma|^2}{TV}. \quad (81)$$

If we let $\omega_\beta\to\infty$ and note that

$$|n_\gamma|^2=\left|\frac{\epsilon_0\chi_{e\gamma}k_\gamma E_\gamma}{e}\right|^2, \quad (82)$$

we obtain the standard high-frequency formula (Bekefi 1966) given by

$$\frac{\partial^2 P}{\partial\Omega\,\partial\omega_\alpha}=\frac{r_0^2 S_\beta V}{\pi}\left[1-\frac{(\mathbf{e}_\beta.\mathbf{k}_\alpha)^2}{k_\alpha^2}\right]\lim_{TV\to\infty}\frac{|n_\gamma|^2}{TV}. \quad (83)$$

The $(\epsilon_\alpha/\epsilon_\beta)^{\frac{1}{2}}$ correction to (83), contained in (80), was derived by Birmingham *et al.* (1965) by a different method.

13. Summary

A general theory for scattering of electromagnetic waves by density fluctuations in a plasma has been presented. The general scattering formula is given by (55), (46), and (50). Its application to incoherent scatter is given by (59), (46), (60), (63)–(65), and to scatter by strongly driven plasma waves by (46) and (80). The theory generalizes previous high-frequency theories, in that it is valid for all frequencies of the incident and scattered electromagnetic waves. It does assume, however, a zero magnetic field, isotropic unperturbed charged particle velocity distribution functions, and the absence of multiple scattering.

An expansion for both incoherent scatter and scattering by strongly driven plasma waves in inverse powers of the frequency ω_β^{-1} of the incident electromagnetic wave has been carried out ((77) and (81), respectively). These expansions show that two types of lower-order corrections must be applied to the high-frequency theory as the incident electromagnetic wave frequency approaches the plasma frequency. The first type of correction is of order $(\omega_p/\omega_\beta)^2$, and must be applied irrespective of the value of the difference frequency ω_γ between the electromagnetic waves. The second correction is of order $(\omega_\gamma/\omega_\beta)^2$, and is clearly of importance only for scattering by the Langmuir waves. These lower-order corrections disappear for the case of backscatter.

As the frequency ω_β of the electromagnetic wave comes closer to ω_p, then of course it is necessary to use the full theory ((59), (46), (60), and (63)–(65), or (46) and (80)). It is important to note that the full theory has non-vanishing higher-order corrections for the backscatter case, even though the lower-order corrections mentioned in the previous paragraph disappear.

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Appendix

In this section we will derive the expressions for the space-time averages given in (56)–(58). Our first step is to derive a relation for the averages in terms of the ensemble average of the Fourier components. According to Parseval's theorem, the average of the product of two variables, $A(\mathbf{r},t)$ and $B(\mathbf{r},t)$, is given by

$$\overline{A(\mathbf{r},t)\,B(\mathbf{r},t)} = \lim_{TV\to\infty} \frac{1}{(2\pi)^4\,TV} \int_{-\infty}^{\infty} A_\gamma B_\gamma^*\, d\omega_\gamma\, d\mathbf{k}_\gamma. \tag{A 1}$$

The ensemble average, on the other hand, is given by

$$\langle A(\mathbf{r},t)\,B(\mathbf{r},t)\rangle = \frac{1}{(2\pi)^8}\int_{-\infty}^{\infty} d\omega_\gamma\, d\omega_{\gamma'}\, d\mathbf{k}_\gamma\, d\mathbf{k}_{\gamma'} \times \langle A_\gamma B_{\gamma'}^*\rangle \exp\left[j(\omega_\gamma-\omega_{\gamma'})t - j(\mathbf{k}_\gamma-\mathbf{k}_{\gamma'}).\mathbf{r}\right]. \tag{A 2}$$

All of the cases studied in this paper have the property that

$$\langle A_\gamma B_{\gamma'}^*\rangle = C_\gamma(AB^*)\,\delta(\omega_\gamma-\omega_{\gamma'})\,\delta(\mathbf{k}_\gamma-\mathbf{k}_{\gamma'}), \tag{A 3}$$

where $C_\gamma(AB^*)$ is some function of A_γ and B_γ^*; therefore

$$\langle A(\mathbf{r},t)\,B(\mathbf{r},t)\rangle = \frac{1}{(2\pi)^8}\int_{-\infty}^{\infty} d\omega_\gamma\, d\mathbf{k}_\gamma\, C_\gamma(AB^*). \tag{A 4}$$

Equating the two averages given by (A 1) and (A 4) yields the desired relation

$$\lim_{TV\to\infty} \frac{1}{TV} A_\gamma B_\gamma^* = \frac{1}{(2\pi)^4} C_\gamma(AB^*). \tag{A 5}$$

Equation (32) shows that

$$C_\gamma[f_u(\mathbf{v})f_u^*(\mathbf{v}')] = (2\pi)^5\,\delta(\mathbf{v}-\mathbf{v}')\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v})f_0(\mathbf{v}). \tag{A 6}$$

Substituting this into (A 5), with A and B equal to $f_u(\mathbf{v})$ and $f_u(\mathbf{v}')$, respectively, yields (56) immediately.

To prove (57) and (58), we shall need to use the linearized Poisson equation

$$\epsilon_\gamma k_\gamma E_\gamma = \frac{je}{\epsilon_0}\left[\int_{-\infty}^{\infty} f_{u\gamma e}(\mathbf{v})\,d\mathbf{v} + \int_{-\infty}^{\infty} f_{u\gamma i}(\mathbf{v})\,d\mathbf{v}\right]. \tag{A 7}$$

If we multiply (A 7) by $f_{u\gamma' e}^*(v')$, take the ensemble average, and assume that the ion and electron motions are uncorrelated, we obtain

$$\epsilon_\gamma k_\gamma \langle f_{u\gamma' e}^*(v')\,E_\gamma\rangle = \frac{je}{\epsilon_0}\int_{-\infty}^{\infty}\langle f_{u\gamma' e}^*(v')f_{u\gamma e}(v)\rangle\,dv. \tag{A 8}$$

Substituting (32) shows that

$$C_\gamma[Ef_{ue}^*(v)] = \frac{(2\pi)^5 je}{\epsilon_0\epsilon_\gamma k_\gamma} f_{0e}(\mathbf{v})\,\delta(\omega_\gamma - \mathbf{k}_\gamma.\mathbf{v}). \tag{A 9}$$

Substituting this result into (A 5), with A and B equal to E and f_{ue}^*, respectively, yields (57).

If we multiply (A 7) by its complex conjugate, and take the ensemble average, we obtain

$$\langle E_\gamma E_{\gamma'}^*\rangle = \frac{e^2}{k_\gamma k_{\gamma'}\epsilon_0^2\epsilon_\gamma\epsilon_{\gamma'}^*}\left[\int_{-\infty}^{\infty}\langle f_{ue\gamma}(\mathbf{v})f_{ue\gamma'}(\mathbf{v}')\rangle\,d\mathbf{v}\,d\mathbf{v}' + \int_{-\infty}^{\infty}\langle f_{ui\gamma}(\mathbf{v})f_{ui\gamma'}(\mathbf{v}')\rangle\,d\mathbf{v}\,d\mathbf{v}'\right]. \tag{A 10}$$

Substituting (32) allows us to determine $C_\gamma(EE^*)$. If this is in turn substituted into (A 5), with A and B equal to E, we obtain (58).

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